

Legislative Redistricting -- Compactness and Population Density Fairness

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Abstract

This article discusses three measures proposed for evaluating the fairness and convenience of legislative redistricting plans: (1) Geographic compactness, (2) population compactness, and (3) a new population density fairness measure. There are over a dozen proposed competing measures of geographic compactness. Pictorial counterexamples demonstrate how most of these measures are unreliable. The isoperimetric quotient is recommended for measuring area compactness because it has a maximum value of one (1) when the district is as compact as a circle, a minimum value approaching zero, and enables direct comparison of any two districts' compactness regardless of size. On the other hand, population compactness helps to ensure districts are convenient for voters and politicians. Population compactness can be measured using the distance of a district's census blocks, weighted by its proportion of the district's population to the district's population centroid. However, due to population distribution patterns, neither area nor population compactness guarantee proportionally fair representation.

To measure whether a legislative redistricting plan fairly represents both urban and rural dwellers' in proportion to their numbers, this article introduces a population density fairness (PDF) measure for evaluating proportional fairness of districts having diverse population densities. A responsive approach for drawing districts considers population equity, preservation of geographic, neighborhood, and political boundaries, proportional fairness, compactness, and administrative ease.

NOTE FROM AUTHOR: The pictorial examples in this draft need to be redrawn for clarity.

¹ The idea for this research paper originated during a redistricting forum at Rockefeller Institute in Albany, New York.

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Background

Legislative districts in the United States are redrawn following every decennial census. The process of reapportionment involves complex issues. The constitution requires districts having roughly equal population. Administrative practicalities demand respect for political and geographic boundaries. All legislative districts in the U.S. smaller than entire states will be redrawn on the basis of the 2010 census. Politicians are prone to use redistricting to concentrate voters of opposing parties in strangely shaped districts so there will be more districts in which their party's voters are a safe, but smaller majority. Or politicians may reward or punish legislators by creating safe districts or by moving a representative's supporters out of his district. Safe districts, it has been plausibly argued tend to be represented by Democrats on the left or Republicans on the right of their own parties. [for example, see Theriault, Sean M. 2004]. Gerrymandering may thus be a factor contributing to what many see as an increasingly polarized House of Representatives [reference needed]. If districting plans are required to be compact it is considered more difficult to achieve these ends. Population compactness of districts also reduce travel time of legislators within their districts and, if existing political boundaries are taken into consideration, increase the shared interests of constituents in each district and presumably provide for legislative representation more attuned to voters' concerns. However, as is shown in this paper, compactness does not ensure fair proportional representation for urban and rural dwellers.

Thirty-five states require legislative districts be compact, believed to make gerrymandering – designing legislative districts so as to advantage one political party – more difficult. Area compactness is readily distinguished by the human eye.ⁱ Dictionary definitions for compact include “arranged within a relatively small space” and “designed to be economical in operation”.ⁱⁱ

UNDER CONSTRUCTION – more introductory text regarding population compactness, mathematical analysis of proportionate partisan fairness of redistricting plans, and proposed measures of population density variance for redistricting purposes is needed here.

I. Geographic Compactness

MacEachren (1985) provides a summary of eleven proposed measures of shape compactness and concludes, “indices based on dispersion of elements of area will provide the most accurate measures of compactness due to their consideration of the unit as a whole.”ⁱⁱⁱ However MacEachren’s sample consisted of shapes of US counties rather than gerrymandered districts. MacEachren’s index, devised to measure a shape’s compactness, assumed that the distribution of characteristics within geographic areas varies at a constant linear rate of change and that compactness should thus be measured as the standard deviation of randomly assigned distribution of values that change at a constant rate spread out over $1/10^{\text{th}}$ area square units. However, the shapes of gerrymandered legislative districts under the preclearance clause of the Voting Rights Act, and ethnic and economic settlement patterns vary sharply at a road or river boundary suggesting the assumption of constant rate of change in constituent characteristics do not hold. In addition, measuring compactness in terms of dispersion of area from a shape’s centroid assigns equal numerical value to shapes having widely differing visual compactness. It would be more difficult to administer and serve constituents in districts drawn using “compactness” measures that assign the same value to geographically disparate districts, as shown in example six below.^{iv} The Blair & Biss method that MacEachren touts is numerically not well-defined in cases of extremely gerrymandered shapes and is difficult to calculate.

Prior articles evaluating various measures of compactness have performed *inter*-measure comparisons, comparing various proposed measures of compactness with one other for the same shapes. This article takes a different approach, performing a comparison of each measure with itself on

different shapes. To be internally consistent, a measure of compactness will not assign an equal numerical value of compactness to two shapes having obviously unequal compactness, especially to two districts having the same area. For ten of the proposed measures of compactness, I show counterexamples to reliability by showing two different geographic shapes having clearly different compactness that are assigned the exact same measure of compactness.

A. Measures That Fail To Reliably Measure Geographic Compactness

The following counterexamples show how nine measures that allege to measure two-dimensional geographic compactness assign the same measurement to shapes that differ in compactness and are therefore not effective, reliable measures of geographic compactness. [NOTE: The following drawings need to be redone for clarity and precision.]

1. The diameter of the largest inscribing circle divided by the diameter of the smallest circumscribing circle (Haggett 1966). A counterexample shows equal inscribing and circumscribing circles, so equal ratios of diameters of those circles, but districts differing in compactness.



Figure 1

2. The diameter of a circle of equal area divided by the diameter the circumscribing circle (Schumm 1956). A counterexample shows two districts with equal areas and equal circumscribing circles but differing in compactness.

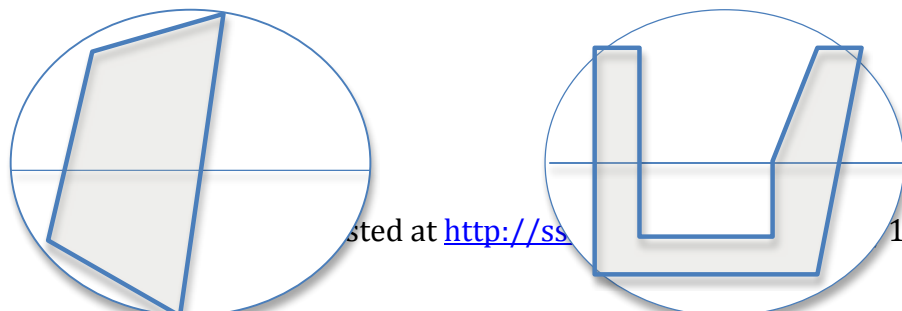


Figure 2

3. The area of intersection of the object and circle of equal area divided by the area of the union of the object and a circle of equal area (Lee & Sallee 1970). The counterexample shows two districts with equal areas and equal ratios of areas inside and outside a circle of equal area, but with differing compactness.



Figure 3

4. The ratio of the longest axis to the shortest axis (Boyce & Clark 1964). The counterexample shows two districts with equal long and short axes, but with differing compactness.

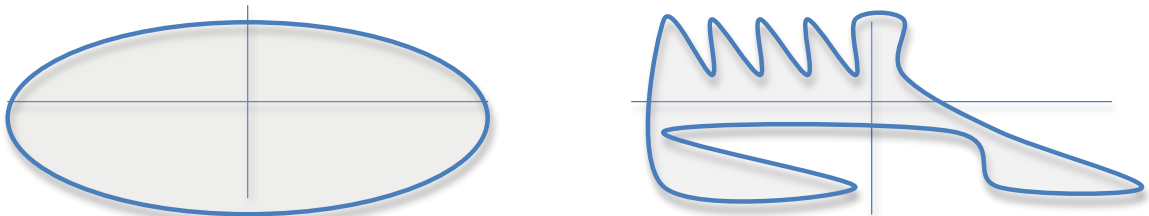


Figure 4

5. The variance in the length of radials extending outward from the object's center (Boyce & Clark 1964). The counterexample shows two districts with the same variance of radials extending outward from the object's center, but with differing compactness.



Figure 5

6. The dispersion of unit of area around the center (Blair & Biss 1967). Measuring compactness in terms of dispersion of area from a shape's centroid, assigns equal numerical values to shapes having differing ease of serving and administering legislative district, as shown below.^v The counterexample shows two districts with equal variance of unit area from their centroids but with different levels of compactness.



Figure 6

Warren Smith aptly expressed the flaws of this method,

The underlying problem is this measure does not care if a district has a really jagged wiggly boundary. For example, a perfect circle, and a circle with very wiggly microscopic structure of the boundary (but still looks like a circle viewed with imperfect vision) have almost exactly the same 'quality.' In fact the wiggly circle has 'better' quality than a perfect square, which is crazy. Why should we care about microscopic stuff? Because it allows a lot of easy manipulation. If I draw a sensible district, then go house by house near its boundary excluding all the Republicans by adding little wiggles, voila. This measure allows that, or anyhow is very insensitive to it. In fact, all the measures in your examples 1-6 [above] allow evilly adding tons of wiggles to all the boundaries with little or no alteration in the flawed 'quality measures.'²

² In a private email communications from Warren Smith to Kathy Dopp, Thu, Jun 9, 2011 at 12:07 PM.

7. Minimizing sum of perimeters. This counter-example shows a districting plan with 3 districts with equal sum of perimeters and thus the same measure, but at least one of the districts in the example at right is less compact than the corresponding district at left. Mutual perimeters could thus be bent to include or exclude certain blocks of voters around the edges of any districts.



Figure 7

8. The ratio of the area to the area of the smallest circumscribing circle (Ehrenberg 1892). The counter-example shows a more compact district (at left) and less compact district (at right) having the exact same measure.



Figure 8

9. Convexity-based measures (Chambers and Miller; Hodge, Marshall, and Patterson). The following two district shapes both have convexity measure of 1 because they are both perfectly convex, yet the district shape at left is more compact. Using convexity as a measure of compactness, a city that is primarily composed of one political party’s members could be cut up in a pizza-pie-like redistricting plan to ensure that the city’s inhabitants did not have proportional representation. Such a gerrymandered plan would have a near perfect measure of compactness using the convexity definition of compactness.

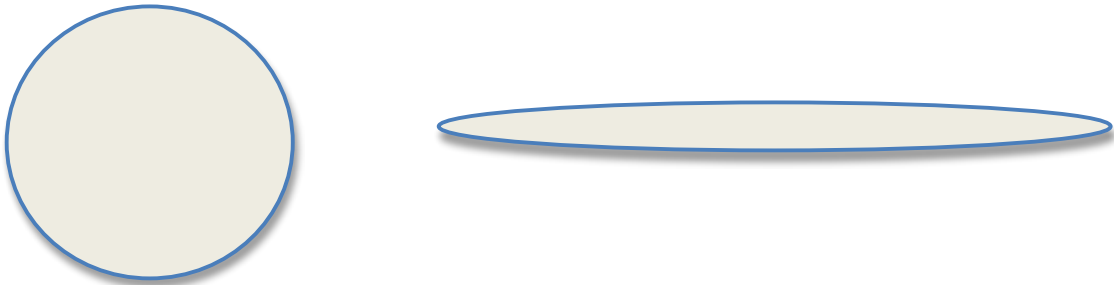
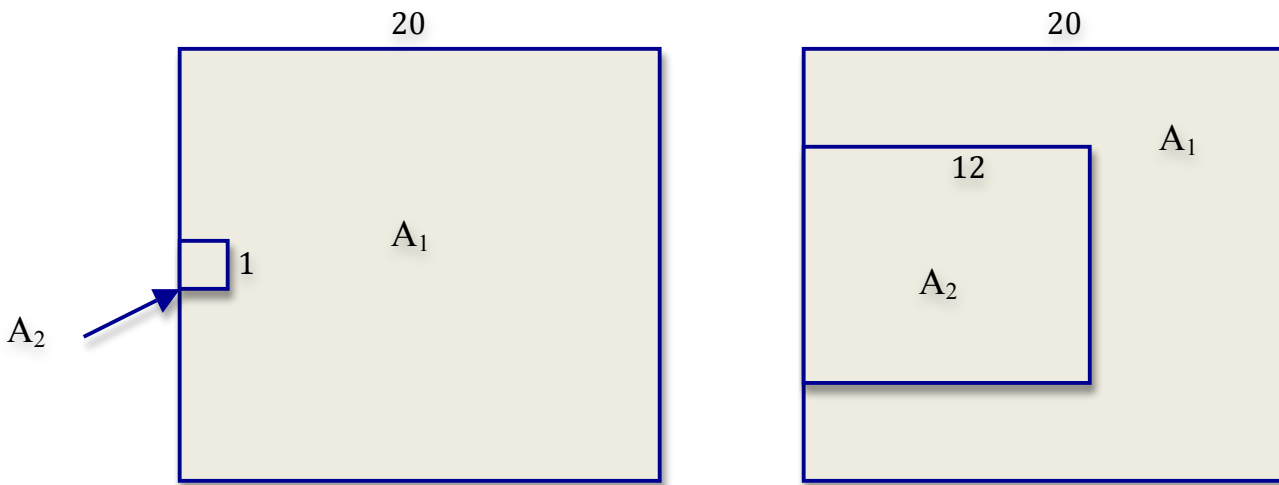


Figure 9

10. A tenth counter-example shows why simple area-to-perimeter measures, or their inverses, do not reliably measure area compactness due to their sensitivity to size variation of districts. Minimizing the sum of district perimeter to area ratios, the visually less compact redistricting plan, below right, is measured as more compact than the visually more compact plan having two virtually square-shaped districts on the left.

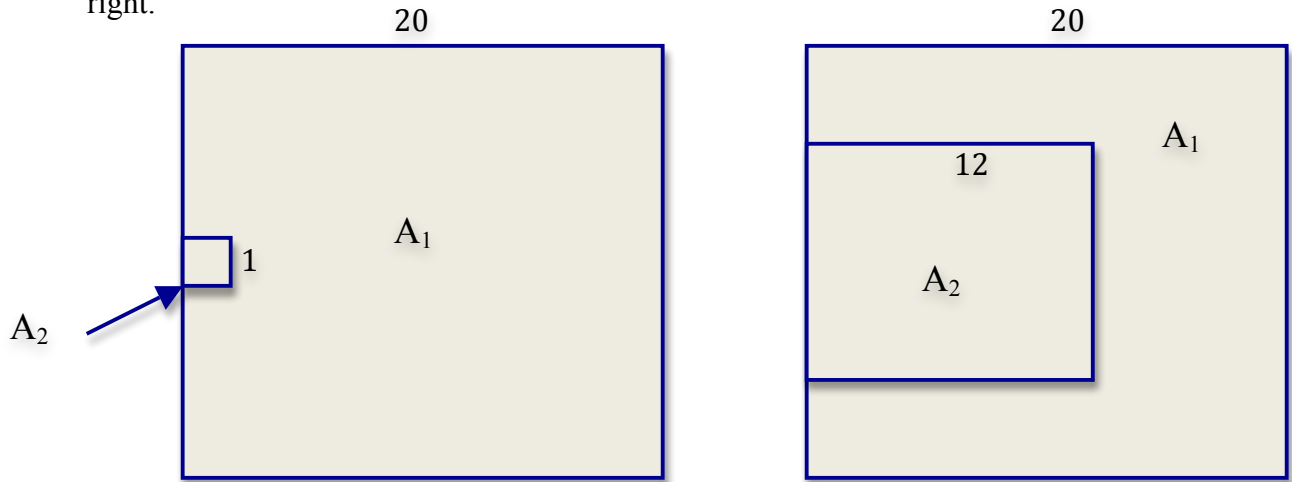


$$\sum_{i=1}^N \frac{P_i}{A_i} = \frac{83}{399} + \frac{4}{1} = 4.21$$

$$\sum_{i=1}^N \frac{P_i}{A_i} = \frac{116}{144} + \frac{48}{256} = 0.99$$

Notice that the “dimensionless” measure $\sum_{i=1}^N \frac{p_i}{A_i^2}$ does *not* rank redistricting plans in the same

order as the measure $\sum_{i=1}^N \frac{p_i}{A_i}$, measuring the left plan below more compact than the plan on the right.



$$\sum_{i=1}^N \frac{p_i}{A_i^2} = \frac{83}{(399)^2} + \frac{4}{1} = 8.16$$

$$\sum_{i=1}^N \frac{p_i}{A_i^2} = \frac{116}{(144)^2} + \frac{48}{(256)^2} = 12.67$$

B. Perimeter-squared-To-Area Measures Of Geographic Compactness

To compare the relative compactness of two individual districts with different areas, the

measure $\frac{A_i}{P_i^2}$ provides the same measure for equally compact districts, regardless of varying district size in area. Any compactness measure is “dimensionless” when the units of measurement cancel from both numerator and denominator. Thus the following measures of district area compactness give a pure

dimensionless number: $\frac{A_i}{P_i^2}, \frac{A_i^{\frac{1}{2}}}{P_i}, \frac{P_i}{A_i^{\frac{1}{2}}}, \text{ or } \frac{P_i^2}{A_i}$. All dimensionless measures of compactness allow direct

comparison of compactness between two shapes having different sizes.

Some proposed compactness measures involving ratios of powers of areas and perimeters are equivalent. For instance, minimizing district perimeter to area ratios is equivalent to maximizing district area to perimeter; multiplying any constant times any ratio of powers of perimeter to area or area to perimeter does not affect the order of how two redistricting plans are ranked. Thus, the measures proposed by several scholars, including Stuart S. Nagel, Tony L. Hill, and Joseph E. Schwartzberg, Daniel D. Polsby, Robert D. Popper, and Avencia, would seem to rank the geometric compactness of any set of competing redistricting plans in the same order.

C. Which Geographic Compactness Measure is Best? The Isoperimetric Quotient

It is a mathematical theorem, called the isoperimetric inequality, that among all two-dimensional regions having a given area, the one with the smallest perimeter is a circle (or equivalently, among all regions with a given perimeter, the circle has the largest area). This result has been known since antiquity, though the first rigorous proof by modern standards is less than 200 years old. More technically: if a closed curve has perimeter P and enclosed area is A , it follows that

$$4\pi A \leq P^2,$$

Moreover, the two sides are equal if and only if the curve is a circle. One can easily check that equality holds for a circle of radius R (whose area is πR^2 and whose circumference is $2\pi R$). For a given closed curve the **isoperimetric quotient** Q is defined as the ratio of its area to the area of a circle having the same perimeter:

$$Q = \frac{4\pi A}{P^2}$$

and the isoperimetric inequality says that $Q \leq 1$ with $Q = 1$ if the shape is a circle.

Our intuitive sense of area compactness means having a relatively large area for a given perimeter. But one must divide the area by the square of the perimeter (A/L^2), not by the perimeter itself, to get a measure that depends only on the shape and not on the absolute size of a region or on the units of measurement (feet, inches meters, etc.)³ While it might seem simpler to define compactness as A/L^2 , without the factor of 4π , introducing this factor gives the index more intuitive content because one is directly comparing the compactness of a given figure to that of a circle (for which $Q = 1$). Some have called the isoperimetric quotient after the names of other scholars who have recommended its usage for measuring compactness of legislative districts. For instance, the developers of the redistricting software available on <http://publicmapping.org> call this compactness measure the Polsby-Popper and Schwartzberg method.⁴

³ Some readers might pause to calculate the ratio A/L (not the square of L) for a square whose side is 1, whose perimeter is therefore 4 and whose area is numerically 1 and then for another rectangle whose side is 12, whose perimeter is 48, and whose area is 144. and whose area. The second rectangle can also be viewed as the first rectangle measured in inches as opposed to feet.

⁴ ‘Developers of the software tout it as an antidote to gerrymandering — the drawing of odd-shaped districts to favor the party in power. "Right now, we don't know what all the options are, because we only see a limited number of plans produced by the political process," said George Mason's [Michael McDonald](#).’ Source: Korte, Gregory, “Software opens up redistricting”. USA TODAY. Updated 3/21/2011.

“Although it has no effect on the optimization, plan-measures that do some kind of average are better than those that simply sum, because the compactness of the whole plan can be discussed on the same scale as the compactness of an individual district, eliminating much confusion.”⁵ Thus, the best measure for comparing redistricting plan compactness is to maximize

the quantity $\frac{1}{n} \sum_{i=1}^n 4\pi \frac{A}{P^2}$.

Due to differences in overall compactness of different states' shapes, the compactness of two different states' plans must be compared by multiplying by the ratios of their overall compactness.

D. A Counter-Example to the Isoperimetric Quotient?

Assume the two shapes below have the exact same isoperimetric quotient, $Q = \frac{4\pi A}{P^2}$ because they have the same perimeter and areas. Is this a counter-example to the reliability of the isoperimetric quotient as a valid measure of compactness? Some may consider the shape on the left to be more “compact” because it fits inside a smaller circumscribed circle. However, if convexity is used as a measure of compactness the figure on the right would be more compact.



http://www.usatoday.com/news/washington/2011-03-20-redistricting_N.htm

⁵ Enders, Robert H., November 2007 INFORMS Annual Meeting.

A redistricting plan having many districts must be considered as a whole by considering the sum or average compactness of all the districts in a state. Conformance to the Voting Rights Act preclearance provisions, geographical boundaries such as mountain ranges, rivers, highways, and to jurisdictional boundaries such as cities, counties, and towns, as well as adjustment to various population distributions come into play. Thus, the isoperimetric quotient measurement of compactness constrains gerrymandering regardless of the fact that sometimes district shapes having the same measure of compactness may fit within different-sized circumscribed circles or be more or less convex. Rotating any phalanges centered on a symmetrical center to be closer together (thus, enabling the overall shape to exist within a smaller circumscribing circle) does not change the overall distances between all points, if travel is constrained within the district. A representative driving within the district to visit all its voters would have the same driving distances overall between points within the district and could not locate a constituent office within the district to be closer overall to more points within the district in either shaped district.

If we were to define a district fitting within a smaller circumscribed circle as more compact than a district fitting within a larger circumscribed circle, then compactness would be a measure of smallness. This would mean that smaller circles would be more compact than larger circles, removing the ability to use our compactness measure to directly compare shapes having unequal areas. A measure of compactness based on smallness would also lack the nice property of assigning values between 0 and 1.

Minimizing the sum of the ratios of perimeter-squared-to-area of districts in a redistricting plan (or maximizing their inverse ratios) reliably finds the most geographically compact legislative redistricting plan. Any constant times the area-to-perimeter-squared measure of compactness ranks the same set of redistricting plans for a given geographic region in exactly the same order.^{vi}

II. Population Compactness

It can be argued that population compactness is a more important feature for legislative districts than geographic compactness. To benefit voters by placing them in districts with population centers relatively close to the places where they live, Robert Enders suggests minimizing the weighted average distance of districts' population to their population centroids. The population

compactness of district j is defined as $\frac{A_j}{\sum_i \|l(x_{ij}) - c_j\|^2 \frac{p(x_{ij})}{P_j}}$ where A_j is its area, and

$\sum_i \|l(x_{ij}) - c_j\|^2 \frac{p(x_{ij})}{P_j}$ is the sum of the squared distances of its census block, or other area

components, to its population centroid. The population centroid, or population center, of each

district j is defined as $c_j = \sum_{x_{ij} \in R_j} l(x_{ij}) \frac{p(x_{ij})}{P_j}$ where $l(x_{ij})$ is a point location, such as the center of

census block or other area component, x_{ij} , within each district and $\frac{p(x_{ij})}{P_j}$ is each component's

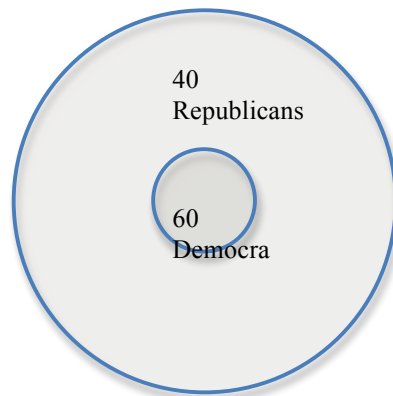
proportion of the total district population, P_j . In other words, a point of each census block determines the district's population centroid in proportion to the census block's population proportion. By using the same distance measure for area and distance to district centroids, this population compactness measure is dimensionless and scale invariant because the area of a district is divided by the square of the average weighted distance from the population to the district centroid.

For uniformly populated regions, this measure is similar to the isoperimetric quotient in that a circular shape has the most population compactness and squares score better than rectangles. Districts that concentrate population near their centers are more compact than those that concentrate population near their edges.⁶

⁶ Enders.

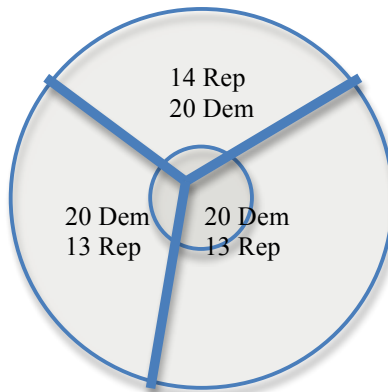
III. Disproportionate Representation, Even with Compactness

Counter-examples show that there can be disproportionate representation, even with compact districts, as measured by $\frac{1}{n} \sum_{i=1}^n 4\pi \frac{A}{P^2}$. The following three examples show how compact districts do not ensure fair proportional representation for citizens. To make the examples simple, we suppose a state is shaped like a circle and has 3 districts; and that 60% of citizens are Democrats who live in a centrally-located city, and 40% are Republicans who live in the rural areas as in figure one below. Let's break this "state" into three equal-population districts in three different ways to see what happens. Since there are three seats and more Democrats than Republicans in our example, the proportionately fairest districts would result in electing two seats with Democrats and one seat with a Republican. In our simplified example, everyone votes and all Republicans live in rural settings and all Democrats live in urban settings.



Example 1: One way to divide the state into 3 districts would be to put equal numbers of Democrats and Republicans into each district. Assuming the radius of the little circle is 1 and the radius of the big circle is three (and the circles are evenly divided by the three lines), the area compactness of

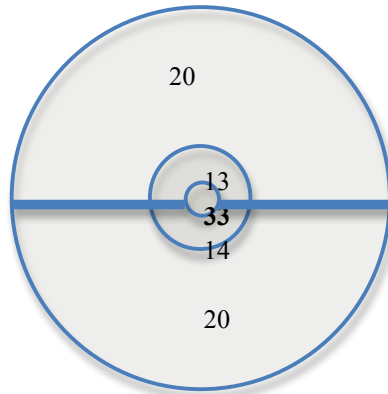
this plan is
$$\frac{4\pi}{n} \sum_{i=1}^n \frac{A_i}{P_i^2} = \frac{3 \cdot 4\pi}{3} \left(\frac{\frac{9}{3}\pi}{\left(6 + \frac{6}{3}\pi\right)^2} \right) = 0.784980709$$



This plan is clearly unfair because all seats would go to Democrats.

Example 2: Another way to divide our circular state would be to create one district of 33 Democrats, one district of 20 Republicans and 13 Democrats and a third district of 20 Republicans and 14 Democrats. Again assuming the radius of the little circle of Democrats is 1, and the radius of the smallest district in its middle is 0.5 and the radius of the big circle is three, the area compactness of this plan is

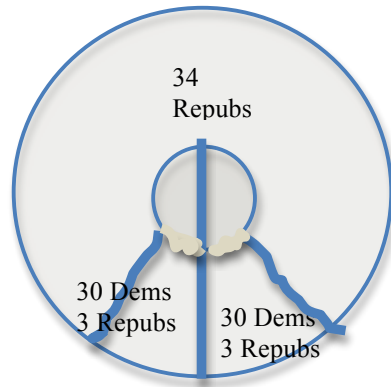
$$\frac{4\pi}{n} \sum_{i=1}^n \frac{A_i}{P_i^2} = \frac{4\pi}{3} \left(\frac{.25\pi}{(\pi)^2} + \left(\frac{\pi(9 - .25)}{(\pi(3 + .5) + 3 - .5)^2} \right) \right) = 0.965546876$$



Clearly this redistricting plan is also unfair because it awards two districts to Republicans and one district to Democrats, even though Democrats outnumber Republicans. Yet, this plan ranks as the most geographically compact plan.

Example 3: This example fairly allocates legislative seats, 2 seats to Democrats and one to a Republican. The radius of the small circle is 1 and the large circle 3. The area compactness of this plan is

$$\frac{4\pi}{n} \sum_{i=1}^n \frac{A_i}{P_i^2} = \frac{4\pi}{3} \left(\frac{((9-1) * 34 * \pi / 40)}{(34(6+2)\pi/40 + 4)^2} + \left(\frac{\pi + 6\pi * (9-1)/40}{(6 * (6+2)\pi/40 + 4)^2} \right) \right) = 0.618652611$$



This plan is the **least** compact of our three examples, and yet is proportionately the fairest of the three!

IV. Population Density Fairness of Redistricting Plans

This section introduces an objective, nonpartisan population density fairness (PDF) measure that can be used to judge how closely a plan awards proportionately fair representation amongst people living in regions with various population densities.

Rural and urban dwellers often have dissimilar political interests. To achieve fair proportionate representation for both, redistricting plans should have districts that are balanced to represent regions with different population density in proportion to their numbers.

Let:

m = the median population density of all districts in plan i ;

d_i = the population density of district i , the number of people in district i divided by its area;

r = A geometric region of a state. Regions could be as detailed as census blocks.

d_r = the population density of region r , the number of people in region r divided by its area;

N = the number of total districts drawn in a state;

p_r = the number of people in region r ;

P = the total number of people living in the state; and

\bar{D} = the weighted mean of state population density where the weights are the relative

proportions of population in each region, $\bar{D} = \sum_r \frac{p_r}{P} d_r = \frac{1}{P} \sum_r p_r d_r$.

Proportional population density fairness requires two conditions to be met. First, the mean population density of districts should be roughly equal to the state's weighted mean population density, weighted by the relative number of people overall living in each density level in the state. If the average density of plan districts is lower than the weighted state average density of its census blocks, the districting plan would tend to disproportionately advantage less populated areas over urban

areas. If the average density of plan districts is higher than the weighted state average, then the plan will tend to advantage more densely populated areas disproportionately. Thus, we

mathematically, minimize $\left| \frac{1}{N} \sum_i d_i - \bar{D} \right|$.

Second, the standard deviation of districts' median densities from the overall median of the districts must be roughly equal to the districts' median itself. If the variance of the district population densities from the median density is greater than the median density, the minority group will be disproportionately advantaged above their proportionate level in the population. If the variance of the district population densities is too small, the majority group will be disproportionately advantaged. For example, if the variance is too large, and the number of rural dwellers is less than the number of urban dwellers, the plan will disproportionately advantage rural voters.

Thus, mathematically, we minimize $\left| m - \left(\frac{1}{N} \sum_i (d_i - m)^2 \right)^{\frac{1}{2}} \right|$. The reason medians, rather than means,

are used in the calculations is because the median voter plus one determines the winning candidate and winning political party of plurality elections. Too-small a standard deviation of districts from the median state population density would award *all* legislative seats to the political party favored by a majority of voters; and, too-large a standard deviation from the overall state median population density would award legislative seats disproportionately to a political party favored by a minority of voters.

Redistricting plans meeting both of the above conditions will most fairly award legislative seats proportionately to urban and rural dwellers. Thus, a redistricting plan population density fairness (PDF) measure is:

$$PDF = \left| m - \left(\frac{1}{N} \sum_i (d_i - m)^2 \right)^{\frac{1}{2}} \right| + \left| \frac{1}{N} \sum_i d_i - \bar{D} \right|$$

The PDF measure sums the absolute values of the differences between: (1) each districts' median population density and the standard deviation of its districts' population densities from its districts' median population density, and (2) a plan districts' mean population density to the state's mean population density weighted by the population proportion of its various regions. The closer this PDF measure is to zero, the proportionately fairer the plan will be with respect to representing regions with diverse population densities.

This new PDF measure has the advantage of being nonpartisan and yet responds to the fact that partisanship of districts is often correlated with district population density. Similar measures could be used to evaluate the proportional fairness of a redistricting plan with respect to how fairly it represents groups having other varying properties.

Let's now use this measure to evaluate the same three examples from section III above to demonstrate how this new measure works to evaluate the proportional fairness of districts' urban versus rural balance.

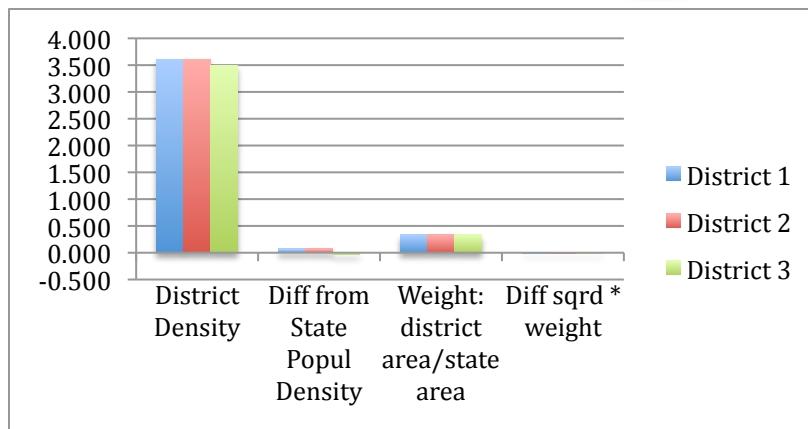
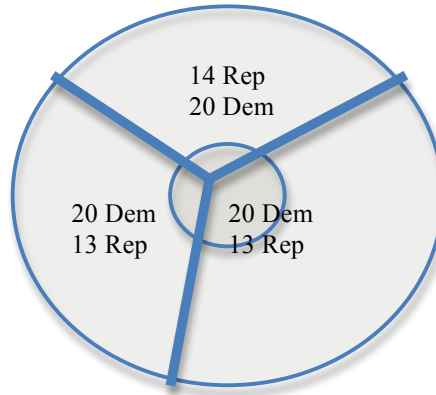
For all three examples of redistricting plans for our hypothetical circular state, the state density is 3.537.

THE EXAMPLES IN THIS SECTION ARE UNDER CONSTRUCTION (Still need to have the calculations explicitly shown.)

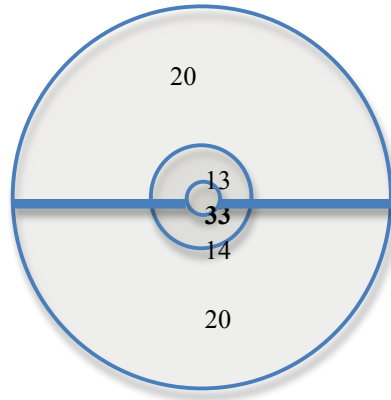
STATE AREA	28.274	sq miles
STATE POPUL	100	people
STATE POP- DENS	3.537	people/sq. mi

For simplicity, the following three examples assume that Democrats live uniformly in more urban areas and Republicans live in more rural areas. There are 60 Democrats and 40 Republicans and 3 districts. Thus, the most fairly proportional redistricting plan would award two districts to Democrats and one to Republicans in this hypothetical circular state.

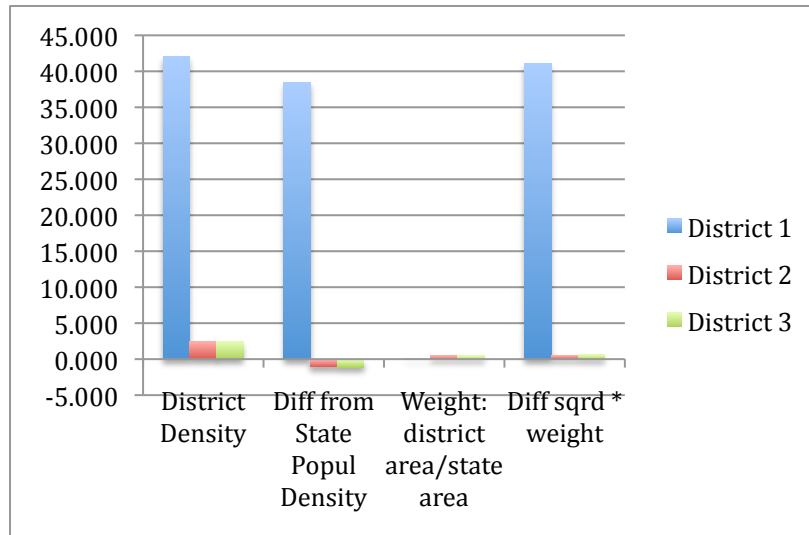
In **Example 1**, three districts with equal population density disproportionately award legislative seats unfairly to the majority, or urban dwellers. Notice that there is zero variation of population densities in the three districts.



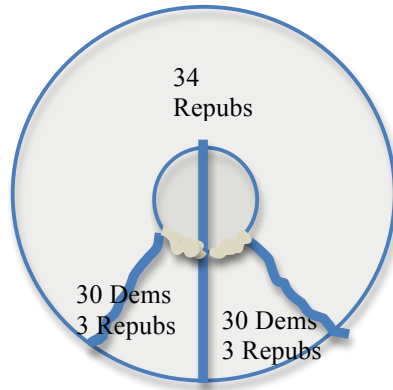
In **Example 2**, one densely populated district of 33 Democrats and two districts with equal population density having 13 Democrats and 20 Republicans each: This plan disproportionately awards legislative seats to the minority political party, or rural voters. Notice, there is too high a variance of the districts from the overall state population density weighted by population.



we find that the PDF is

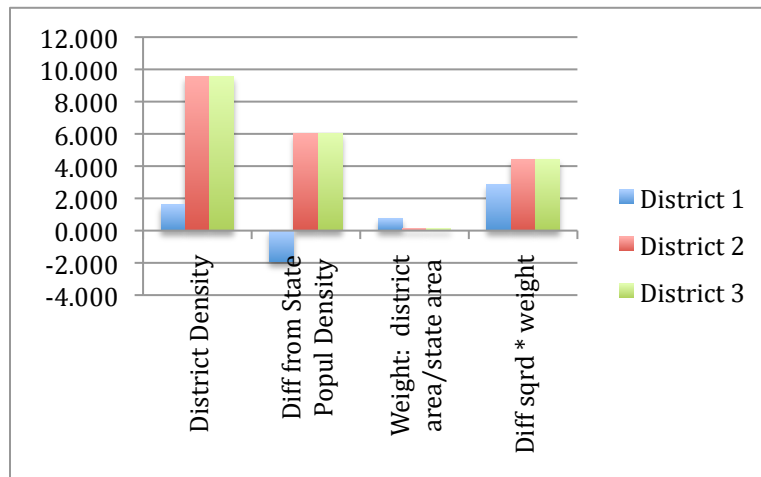


In **Example 3**, one densely populated district and two districts with equal population density: This redistricting plan is the fairest of the three in that it awards seats proportionately most fairly to urban and rural dwellers. Notice that there is adequate variance among the three districts' population densities – closer to the median population density in this proportionately fairer plan.



The calculated PDF is

The districts' population densities are shown in the chart below:



V. The Legislative Redistricting Context – Combining the Important Factors

In drawing district maps, compactness comes into play only after other considerations are met. 'Communities of interest' should be preserved as much as possible and “neighborhoods mean a lot in redistricting ... neighborhoods which would probably not pass a compactness test. Our streets are said to follow the windings of cow paths and our neighborhoods are fairly unruly in shape.”⁷ A state district map could be chosen that:

- (i) has equal population districts to within 0.5%;
- (ii) utilizes natural and geographic boundaries and barriers such as vehicular impassable mountain ranges and rivers in the creation of district boundaries;
- (iii) utilizes existing neighborhood, municipal, and county government boundaries and preserves the requirements of the Voting Rights Act in the creation of district boundaries; and
- (iv) has a population density fairness close to one to ensure roughly proportional representation for regions having diverse population densities.⁸

One possible procedure for redistricting, might be to allow various parties and citizens' groups to propose district maps that meet the above four conditions, and the "winning" map would be one having:

- (a) the most geographic or population compactness, and
- (b) the least number of average election jurisdictions per district (to keep election administrative simple). These last two conditions could be equally weighted.

⁷ Email exchange of Sat Jun 4, 2011 on the email discussion list LWVTopics@yahoogroups.com. The quote is by Hollie Courage, former President of the League of Women Voters of Rhode Island (LWV, RI). Barbara Klein of the LWV, Arizona reminded me during the same conversation of the need to preserve 'communities of interest'. Please see <http://www.lwvri.org/redistricting.htm> and “The Law and Drawing District Lines” <http://www.lwvri.org/law-drawingdistrictlines.htm>.

⁸ Some of the language in (ii) and (iii) is borrowed from Roz McGee, a Utah House member who made a different redistricting proposal in the 2007 Utah legislative session.

Notice the first four conditions listed above are considered to be more crucial and yet the second two conditions, compactness and administrative ease, are important for the ease and cost of serving, being served, and administering legislative districts.

Such a process for deciding on which district map to adopt might reduce the need for subjective judgments or competing claims and create districts to better serve voters, and be easier for representatives to serve and for election officials to administer.

Conclusion

This paper has shown that most formerly proposed measures of area compactness for purposes of legislative redistricting are unreliable, and that area compactness of redistricting plans can be reliably measured using the isoperimetric quotient — the ratio of the area of the district to the area of a circle having the same perimeter. It also discusses a population compactness measure of convenient proximity within districts for voters and representatives.

Redistricting plans having more compactness may be proportionately less fair than plans having less compactness. On the other hand, a new population density fairness (PDF) measure evaluates the proportional fairness of reapportionment plans in representing persons living in regions of diverse population densities. Plans with PDF values closer to zero (0) more fairly represent urban and rural dwellers in proportion to their numbers. The efficacy of the PDF measure requires empirical testing.

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Malcolm Sherman, Professor of Mathematics at State University of New York at Albany, provided many helpful inputs, including a convincing argument for why the isoperimetric quotient is the best choice among the equivalent measures of compactness, the alleged counterexample to the isoperimetric quotient as a measure of compactness, and suggested rewrites of several sections. Robert Enders thankfully caught a key logic error and several oversights in my original paper; provided an example showing that the simpler area-to-perimeter compactness measures are not equivalent to the dimensionless, scale-invariant measures of area compactness, introduced me to his proposed measure for population compactness, which is included and recommended in this paper, and suggested several articles which will be discussed and cited in this paper's final version. Warren Smith pointed out the vestiges of the same logic error by providing counterexamples to an incorrect claim. Jameson Quinn thankfully found some exceptions, counterexample cases, to an earlier simpler version of my proposed PDF measure, prompting me to test a wider range of cases and adjust the measure. The language in (ii) and (iii) in "The Legislative Redistricting Context" section is borrowed from Roz McGee, a Utah House member who made a different redistricting proposal in the 2007 Utah legislative session.

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ⁱ Hill (2009)

ⁱⁱ <http://dictionary.com>

ⁱⁱⁱ Maceachen, *Compactness of Geographic Shape: Comparison and Evaluation of Measures* (1985)

^{iv} An oval, which is easy to administer and for people to know when they are in and outside of a district has equal Blair and Biss measure with a district that looks like a many-petalled flower where the candidate or legislator would be constantly driving in and out of his district and have a lot of trouble knowing when he was in or out of it. Not only do the assumptions of constant rate of change of the distribution of like interests for constituents not hold, but it would be very difficult to administer and to serve one's constituents using a measure that gives the exact same measure for such geographically different districts.

^v An oval, which is easy to administer and for people to know when they are in and outside of a district has equal Blair and Biss measure with a district that looks like a many-petalled flower where the candidate or legislator would be constantly driving in and out of his district and have a lot of trouble knowing when he was in or out of it. Not only do the assumptions of constant rate of change of the distribution of like interests for constituents not hold, but it would be very difficult to administer and to serve one's constituents using a measure that gives the exact same measure for such geographically different districts.

^{vi} To standardize the units in numerator and denominator, make both either linear (one dimensional) as in $\frac{A_i^{\frac{1}{2}}}{P_i}$ or make both

2-dimensional as in $\frac{A_i}{P_i^2}$ or $\frac{P_i^2}{A_i}$. A similar measure for three-dimensional compactness could use ratios or inverse ratios of volume to surface area.